積分８題

ー$1-x^2$, $1+x^2$ を含む積分ー

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積分①

$$\int \frac{1}{1-x^2}\mathrm{d}x = \int \frac{1}{(1+x)(1-x)}\mathrm{d}x = \int \frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\mathrm{d}x$$

$$= \frac{1}{2}(\log|1+x| - \log|1-x|) + C$$ （C は積分定数）

$$= \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

$|x|<1$ のとき

$$\int \frac{1}{1-x^2}\mathrm{d}x = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

$$= \tanh^{-1} x + C$$

$-1 < \tanh t < 1$ より、$|x| < 1$ のとき

$x = \tanh t = \dfrac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}} = \dfrac{\mathrm{e}^{2t}-1}{\mathrm{e}^{2t}+1}$ とおくと

$$x(\mathrm{e}^{2t}+1) = \mathrm{e}^{2t} - 1$$

$$1 + x = \mathrm{e}^{2t}(1-x)$$

$$\mathrm{e}^{2t} = \frac{1+x}{1-x} \qquad \therefore t = \frac{1}{2}\log\frac{1+x}{1-x}$$

積分②

$x = \tan\theta$ とおくと、$\mathrm{d}x = \dfrac{1}{\cos^2\theta}\mathrm{d}\theta$ より

$$\int \frac{1}{1+x^2}\mathrm{d}x = \int \frac{1}{1+\tan^2\theta}\cdot\frac{1}{\cos^2\theta}\mathrm{d}\theta = \int \frac{1}{\cos^2\theta+\sin^2\theta}\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta = \theta + C \quad （C \text{ は積分定数}）$$

$$= \tan^{-1} x + C$$

積分③

$x = \sin\theta$ とおくと、$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ より

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta = \int \frac{1}{\cos\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \int \mathrm{d}\theta = \theta + C$$ （C は積分定数）

$$= \sin^{-1}x + C$$

積分④

$$1+\sinh^2 t = 1+\left(\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}\right)^2 = \left(\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\right)^2 = \cosh^2 t$$

$x = \sinh t = \dfrac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}$ とおくと、$\mathrm{d}x = \dfrac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\mathrm{d}t = \cosh t\,\mathrm{d}t$ より

$$\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{1+\sinh^2 t}}\cosh t\,\mathrm{d}t = \int \frac{1}{\cosh t}\cosh t\,\mathrm{d}t = \int \mathrm{d}t$$

$$= t + C$$ （C は積分定数）

$$= \sinh^{-1} x + C$$

$$= \log\left(x+\sqrt{1+x^2}\right) + C$$

$x = \sinh t = \dfrac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}$ より

$$2x = \mathrm{e}^t - \mathrm{e}^{-t}$$

$$\mathrm{e}^t - 2x - \mathrm{e}^{-t} = 0$$

$$\mathrm{e}^{2t} - 2x\mathrm{e}^t - 1 = 0$$

$\mathrm{e}^t > 0$ だから

$$\mathrm{e}^t = x + \sqrt{1+x^2}$$

積分⑤

$x = \sin\theta$ とおくと、$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ より

$$\int \sqrt{1-x^2}\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta = \int \cos\theta\cdot\cos\theta\,\mathrm{d}\theta$$

$$= \int \cos^2\theta\,\mathrm{d}\theta = \int \frac{\cos 2\theta + 1}{2}\mathrm{d}\theta$$

$$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$$ （C は積分定数）

$$= \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C = \frac{1}{2}(\sin\theta\cos\theta + \theta) + C$$

$$= \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1}x\right) + C$$

積分⑥

$$1+\sinh^2 t = 1+\left(\frac{e^t - e^{-t}}{2}\right)^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 = \cosh^2 t$$

$x = \sinh t = \dfrac{e^t - e^{-t}}{2}$ とおくと、$dx = \dfrac{e^t + e^{-t}}{2} dt = \cosh t \, dt$ より

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+\sinh^2 t}\cosh t\,dt = \int \cosh t \cdot \cosh t\,dt$$

$$= \int \cosh^2 t\,dt = \int \left(\frac{e^t + e^{-t}}{2}\right)^2 dt$$

$$= \int \frac{e^{2t} + e^{-2t} + 2}{4} dt = \int \left(\frac{e^{2t} + e^{-2t}}{4} + \frac{1}{2}\right) dt$$

$$= \frac{e^{2t} - e^{-2t}}{8} + \frac{1}{2}t + C$$ （C は積分定数）

積分⑥

$$= \frac{(\mathrm{e}^{t} - \mathrm{e}^{-t})(\mathrm{e}^{t} + \mathrm{e}^{-t})}{8} + \frac{1}{2}t + C$$

$$= \frac{1}{2}\frac{\mathrm{e}^{t} - \mathrm{e}^{-t}}{2}\frac{\mathrm{e}^{t} + \mathrm{e}^{-t}}{2} + \frac{1}{2}t + C$$

$$= \frac{1}{2}\sinh t \cosh t + \frac{1}{2}t + C = \frac{1}{2}(\sinh t \cosh t + t) + C$$

$$= \frac{1}{2}\left(x\sqrt{1 + x^2} + \sinh^{-1} x\right) + C$$

$$= \frac{1}{2}\left\{x\sqrt{1 + x^2} + \log\left(x + \sqrt{1 + x^2}\right)\right\} + C$$

積分⑦

$x = \sin\theta$ とおくと、$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$ より

$$\int \frac{1}{(1-x^2)^{3/2}}\mathrm{d}x = \int \frac{1}{(1-\sin^2\theta)^{3/2}}\cos\theta\,\mathrm{d}\theta = \int \frac{1}{\cos^3\theta}\cdot\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{\cos^2\theta}\mathrm{d}\theta = \tan\theta + C \quad （C \text{ は積分定数}）$$

$$= \frac{\sin\theta}{\cos\theta} + C = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

積分⑧

$x = \tan\theta$ とおくと、$\mathrm{d}x = \dfrac{1}{\cos^2\theta}\mathrm{d}\theta$ より

$$\int \frac{1}{(1+x^2)^{3/2}}\mathrm{d}x = \int \frac{1}{(1+\tan^2\theta)^{3/2}}\frac{1}{\cos^2\theta}\mathrm{d}\theta = \int \cos^3\theta \cdot \frac{1}{\cos^2\theta}\mathrm{d}\theta$$

$$= \int \cos\theta\,\mathrm{d}\theta = \sin\theta + C$$ （C は積分定数）

$$= \tan\theta \cdot \cos\theta + C = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + C$$

$$= \frac{x}{\sqrt{1+x^2}} + C$$

まとめ

積分定数を C として

$$\int \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

$$= \tanh^{-1} x + C \quad （|x| < 1 \text{ のとき}）$$

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1} x + C$$

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \sinh^{-1} x + C = \log\left(x + \sqrt{1+x^2}\right) + C$$

まとめ

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1} x\right) + C$$

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1+x^2} + \sinh^{-1} x\right) + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x + \sqrt{1+x^2}\right)\right\} + C$$

$$\int \frac{1}{(1-x^2)^{3/2}}\,\mathrm{d}x = \frac{x}{\sqrt{1-x^2}} + C$$

$$\int \frac{1}{(1+x^2)^{3/2}}\,\mathrm{d}x = \frac{x}{\sqrt{1+x^2}} + C$$

付録：積分①の別解

$-1 < x < 1$ において、$x = \tanh t = \dfrac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}}$ とおくと

$$1 - x^2 = \frac{(\mathrm{e}^t + \mathrm{e}^{-t})^2 - (\mathrm{e}^t - \mathrm{e}^{-t})^2}{(\mathrm{e}^t + \mathrm{e}^{-t})^2} = \frac{4}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}$$

$$\mathrm{d}x = \frac{(\mathrm{e}^t + \mathrm{e}^{-t})^2 - (\mathrm{e}^t - \mathrm{e}^{-t})^2}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}\mathrm{d}t = \frac{4}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}\mathrm{d}t$$

より

$$\int \frac{1}{1 - x^2}\mathrm{d}x = \int \frac{(\mathrm{e}^t + \mathrm{e}^{-t})^2}{4} \cdot \frac{4}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}\mathrm{d}t = \int \mathrm{d}t$$

$$= t + C = \tanh^{-1} x + C \quad （C \text{ は積分定数}）$$

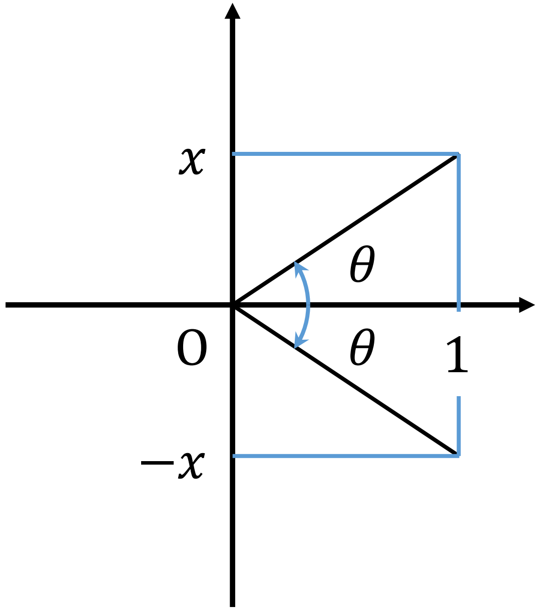
付録：積分②の別解

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \int \frac{1}{(1+ix)(1-ix)}\,\mathrm{d}x = \int \frac{1}{2}\left(\frac{1}{1+ix}+\frac{1}{1-ix}\right)\mathrm{d}x$$

$$= \frac{1}{2i}\left(\log(1+ix) - \log(1-ix)\right) + C \quad （C\text{ は積分定数}）$$

$$= \frac{1}{2i}\log\frac{1+ix}{1-ix} + C$$

$$= \tan^{-1} x + C$$



$$x = \tan\theta = \frac{1}{i}\frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}} = \frac{1}{i}\frac{\mathrm{e}^{2i\theta}-1}{\mathrm{e}^{2i\theta}+1}$$ より

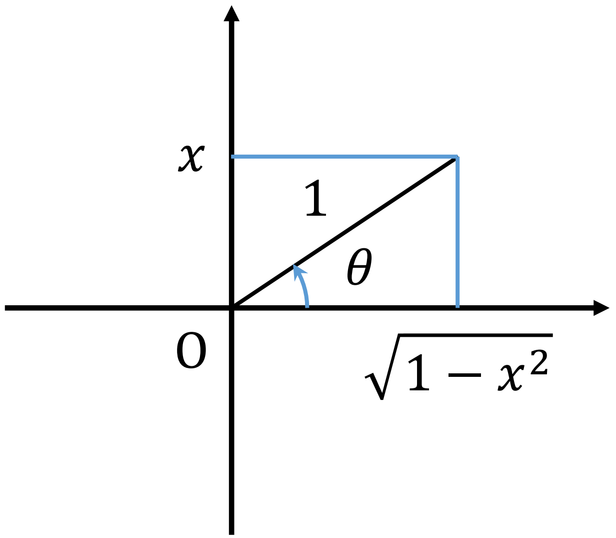
$$ix\left(\mathrm{e}^{2i\theta}+1\right) = \mathrm{e}^{2i\theta}-1$$

$$1+ix = \mathrm{e}^{2i\theta}(1-ix)$$

$$\mathrm{e}^{2i\theta} = \frac{1+ix}{1-ix} \qquad \therefore\ \theta = \frac{1}{2i}\log\frac{1+ix}{1-ix}$$

付録：積分③の補足

$$x = \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ より}$$



$$2ix = e^{i\theta} - e^{-i\theta}$$

$$e^{i\theta} - 2ix - e^{-i\theta} = 0$$

$$e^{2i\theta} - 2ix e^{i\theta} - 1 = 0$$

$$e^{i\theta} = ix \pm \sqrt{-x^2 + 1} = ix \pm \sqrt{1 - x^2}$$

$$\therefore \sin^{-1} x = \theta = \frac{1}{i}\log\left(ix \pm \sqrt{1-x^2}\right) = \frac{1}{i}\log\left(\pm\sqrt{1-x^2} + ix\right)$$

$$= \frac{1}{i} \cdot i \arg\left(\pm\sqrt{1-x^2} + ix\right) = \arg\left(\pm\sqrt{1-x^2} + ix\right)$$

付録：積分③の補足

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{i}\log\left(ix \pm \sqrt{1-x^2}\right)\right] = \frac{1}{i}\cdot\frac{i \mp \dfrac{x}{\sqrt{1-x^2}}}{ix \pm \sqrt{1-x^2}} \quad \text{（複号同順）}$$

$$= \frac{1}{i\sqrt{1-x^2}}\cdot\frac{i\sqrt{1-x^2} \mp x}{ix \pm \sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}\cdot\frac{i\sqrt{1-x^2} \mp x}{-x \pm i\sqrt{1-x^2}}$$

$$= \pm\frac{1}{\sqrt{1-x^2}}$$

付録：積分③の補足

したがって、

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1} x + C \quad （C \text{ は積分定数}）$$

$$= \frac{1}{i}\log\left(ix + \sqrt{1-x^2}\right) + C$$

付録：積分④の検算

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log\left(x+\sqrt{1+x^2}\right)\right]=\frac{1+\dfrac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{x+\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+x^2}}$$

付録：積分④の別解

$$\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x = \frac{1}{i}\int \frac{1}{\sqrt{1-(ix)^2}}i\mathrm{d}x$$

$$= \frac{1}{i}\cdot\frac{1}{i}\log\left(i\cdot ix + \sqrt{1-(ix)^2}\right) + C \qquad (C \text{ は積分定数})$$

$$= -\log\left(-x + \sqrt{1+x^2}\right) + C$$

$$= \log\frac{1}{-x+\sqrt{1+x^2}} + C$$

$$= \log\left(x + \sqrt{1+x^2}\right) + C$$

付録：積分⑤の検算

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$ より

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)\right] = \frac{1}{2}\left(\sqrt{1-x^2} - x\cdot\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{2}\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{2}\left(\sqrt{1-x^2} + \frac{1-x^2}{\sqrt{1-x^2}}\right)$$

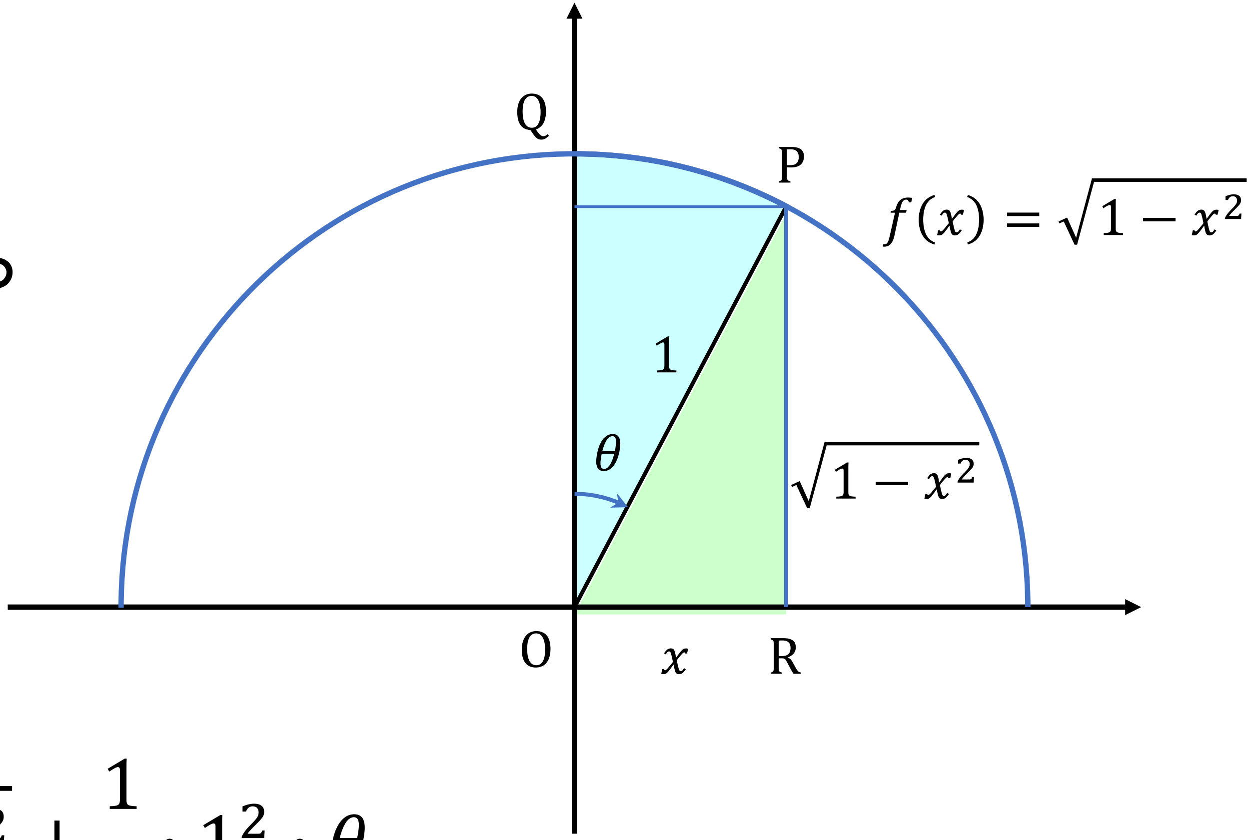
$$= \frac{1}{2}\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right) = \sqrt{1-x^2}$$

付録:積分⑤の補足

曲線 $f(x)=\sqrt{1-x^2}$ は半円だから

$\int_0^x \sqrt{1-x^2}\mathrm{d}x$ は三角形 OPR と

扇形 OPQ の面積の和になる。



したがって、

$$\int_0^x \sqrt{1-x^2}\mathrm{d}x = \frac{1}{2}\cdot x\cdot\sqrt{1-x^2} + \frac{1}{2}\cdot 1^2\cdot\theta$$

$$= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x$$

$$= \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1}x\right)$$

付録：積分⑥の別解

$$\int \sqrt{1+x^2}\mathrm{d}x = \frac{1}{i}\int \sqrt{1-(ix)^2}i\mathrm{d}x$$

$$= \frac{1}{i}\cdot\frac{1}{2}\left\{ix\sqrt{1-(ix)^2} + \frac{1}{i}\log\left(i\cdot ix + \sqrt{1-(ix)^2}\right)\right\} + C$$

（C は積分定数）

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} - \log\left(-x+\sqrt{1+x^2}\right)\right\} + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\frac{1}{-x+\sqrt{1+x^2}}\right\} + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x+\sqrt{1+x^2}\right)\right\} + C$$

付録：積分⑦の検算

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{\sqrt{1-x^2}}\right] = \frac{\sqrt{1-x^2} - x\cdot\frac{-x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{(1-x^2)+x^2}{(1-x^2)\sqrt{1-x^2}} = \frac{1}{(1-x^2)^{3/2}}$$

付録：積分⑧の検算

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{\sqrt{1+x^2}}\right]=\frac{\sqrt{1+x^2}-x\cdot\dfrac{x}{\sqrt{1+x^2}}}{1+x^2}$$

$$=\frac{(1+x^2)-x^2}{(1+x^2)\sqrt{1+x^2}}=\frac{1}{(1+x^2)^{3/2}}$$