

積分6題

— $1 - x^2$, $1 + x^2$ を含む積分 —

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積分①

$$\begin{aligned}\int \frac{1}{1-x^2} dx &= \int \frac{1}{(1+x)(1-x)} dx = \int \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx \\&= \frac{1}{2} (\log|1+x| - \log|1-x|) + C \quad (C \text{ は積分定数}) \\&= \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C \\&= \tanh^{-1} x + C\end{aligned}$$

$$x = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t} - 1}{e^{2t} + 1} \text{ より}$$

$$x(e^{2t} + 1) = e^{2t} - 1$$

$$1 + x = e^{2t}(1 - x)$$

$$e^{2t} = \frac{1+x}{1-x} \quad \therefore t = \frac{1}{2} \log \frac{1+x}{1-x}$$

積分②

$x = \tan \theta$ とおくと、 $dx = \frac{1}{\cos^2 \theta} d\theta$ より

$$\int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2 \theta} \cdot \frac{1}{\cos^2 \theta} d\theta = \int \frac{1}{\cos^2 \theta + \sin^2 \theta} d\theta$$

$$= \int d\theta = \theta + C \quad (C \text{ は積分定数})$$

$$= \tan^{-1} x + C$$

積分③

$x = \sin \theta$ とおくと、 $dx = \cos \theta \, d\theta$ より

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-\sin^2 \theta}} \cos \theta \, d\theta = \int \frac{1}{\cos \theta} \cos \theta \, d\theta \\ &= \int d\theta = \theta + C \quad (C \text{ は積分定数}) \\ &= \sin^{-1} x + C\end{aligned}$$

積分④

$$1 + \sinh^2 t = 1 + \left(\frac{e^t - e^{-t}}{2} \right)^2 = \left(\frac{e^t + e^{-t}}{2} \right)^2 = \cosh^2 t$$

$$x = \sinh t = \frac{e^t - e^{-t}}{2} \text{ とおくと、 } dx = \frac{e^t + e^{-t}}{2} dt = \cosh t dt \text{ より}$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{1}{\sqrt{1+\sinh^2 t}} \cosh t dt = \int \frac{1}{\cosh t} \cosh t dt = \int dt$$

$$= t + C \quad (C \text{ は積分定数})$$

$$= \sinh^{-1} x + C$$

$$= \log \left(x + \sqrt{1+x^2} \right) + C$$

$$x = \sinh t = \frac{e^t - e^{-t}}{2} \text{ より}$$

$$2x = e^t - e^{-t}$$

$$e^t - 2x - e^{-t} = 0$$

$$e^{2t} - 2xe^t - 1 = 0$$

$$e^t > 0 \text{ だから}$$

$$e^t = x + \sqrt{1+x^2}$$

積分⑤

$x = \sin \theta$ とおくと、 $dx = \cos \theta \, d\theta$ より

$$\int \sqrt{1-x^2} dx = \int \sqrt{1-\sin^2 \theta} \cos \theta \, d\theta = \int \cos \theta \cdot \cos \theta \, d\theta$$

$$= \int \cos^2 \theta \, d\theta = \int \frac{\cos 2\theta + 1}{2} d\theta$$

$$= \frac{1}{4} \sin 2\theta + \frac{1}{2} \theta + C \quad (C \text{ は積分定数})$$

$$= \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2} \theta + C = \frac{1}{2} (\sin \theta \cos \theta + \theta) + C$$

$$= \frac{1}{2} \left(x\sqrt{1-x^2} + \sin^{-1} x \right) + C$$

積分⑥

$$1 + \sinh^2 t = 1 + \left(\frac{e^t - e^{-t}}{2} \right)^2 = \left(\frac{e^t + e^{-t}}{2} \right)^2 = \cosh^2 t$$

$$x = \sinh t = \frac{e^t - e^{-t}}{2} \text{ とおくと、 } dx = \frac{e^t + e^{-t}}{2} dt = \cosh t \, dt \text{ より}$$

$$\int \sqrt{1 + x^2} dx = \int \sqrt{1 + \sinh^2 t} \cosh t \, dt = \int \cosh t \cdot \cosh t \, dt$$

$$= \int \cosh^2 t \, dt = \int \left(\frac{e^t + e^{-t}}{2} \right)^2 dt$$

$$= \int \frac{e^{2t} + e^{-2t} + 2}{4} dt = \int \left(\frac{e^{2t} + e^{-2t}}{4} + \frac{1}{2} \right) dt$$

$$= \frac{e^{2t} - e^{-2t}}{8} + \frac{1}{2} t + C \quad (C \text{ は積分定数})$$

積分⑥

$$\begin{aligned} &= \frac{(e^t - e^{-t})(e^t + e^{-t})}{8} + \frac{1}{2}t + C \\ &= \frac{1}{2} \frac{e^t - e^{-t}}{2} \frac{e^t + e^{-t}}{2} + \frac{1}{2}t + C \\ &= \frac{1}{2} \sinh t \cosh t + \frac{1}{2}t + C = \frac{1}{2}(\sinh t \cosh t + t) + C \\ &= \frac{1}{2} \left(x\sqrt{1+x^2} + \sinh^{-1} x \right) + C \\ &= \frac{1}{2} \left\{ x\sqrt{1+x^2} + \log \left(x + \sqrt{1+x^2} \right) \right\} + C \end{aligned}$$

まとめ

積分定数を C として

$$\int \frac{1}{1-x^2} dx = \tanh^{-1} x + C = \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + C$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C = \log \left(x + \sqrt{1+x^2} \right) + C$$

まとめ

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \left(x\sqrt{1-x^2} + \sin^{-1} x \right) + C$$

$$\begin{aligned} \int \sqrt{1+x^2} dx &= \frac{1}{2} \left(x\sqrt{1+x^2} + \sinh^{-1} x \right) + C \\ &= \frac{1}{2} \left\{ x\sqrt{1+x^2} + \log \left(x + \sqrt{1+x^2} \right) \right\} + C \end{aligned}$$

付録: 積分①の別解

$$x = \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} \text{ とおくと}$$

$$1 - x^2 = \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{(e^t + e^{-t})^2} = \frac{4}{(e^t + e^{-t})^2}$$

$$dx = \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{(e^t + e^{-t})^2} dt = \frac{4}{(e^t + e^{-t})^2} dt$$

より

$$\int \frac{1}{1 - x^2} dx = \int \frac{(e^t + e^{-t})^2}{4} \cdot \frac{4}{(e^t + e^{-t})^2} dt = \int dt$$

$$= t + C = \tanh^{-1} x + C \quad (C \text{ は積分定数})$$

付録：積分②の別解

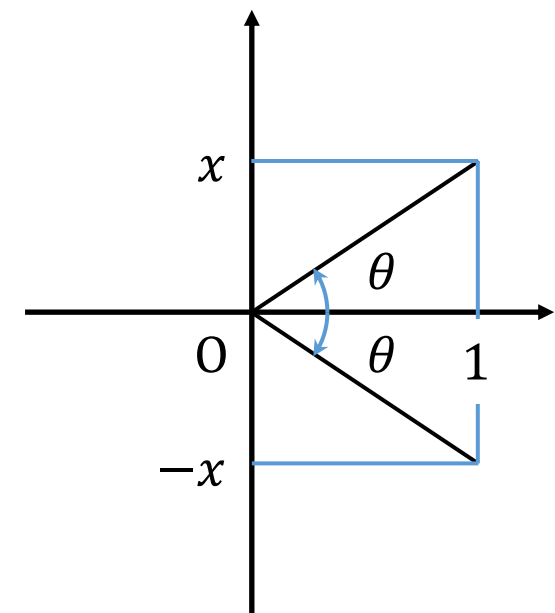
$$\begin{aligned}\int \frac{1}{1+x^2} dx &= \int \frac{1}{(1+ix)(1-ix)} dx = \int \frac{1}{2} \left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) dx \\&= \frac{1}{2i} (\log(1+ix) - \log(1-ix)) + C \quad (C \text{ は積分定数}) \\&= \frac{1}{2i} \log \frac{1+ix}{1-ix} + C \\&= \tan^{-1} x + C\end{aligned}$$

$$x = \tan \theta = \frac{1}{i} \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = \frac{1}{i} \frac{e^{2i\theta} - 1}{e^{2i\theta} + 1} \quad \text{より}$$

$$ix(e^{2i\theta} + 1) = e^{2i\theta} - 1$$

$$1 + ix = e^{2i\theta} (1 - ix)$$

$$e^{2i\theta} = \frac{1+ix}{1-ix} \quad \therefore \theta = \frac{1}{2i} \log \frac{1+ix}{1-ix}$$



付録：積分③の補足

$$x = \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ より}$$

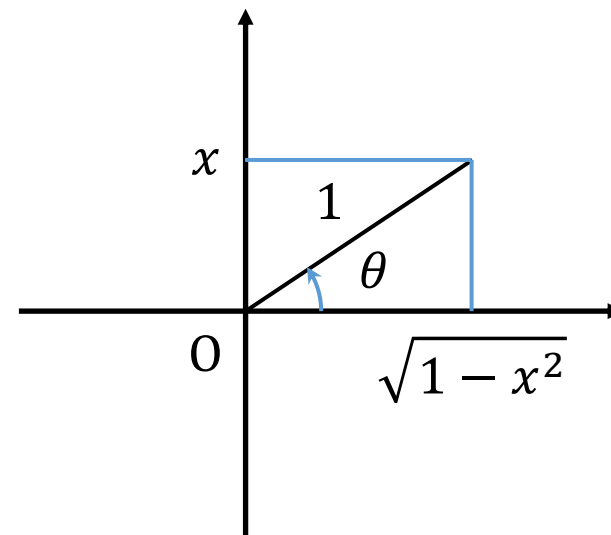
$$2ix = e^{i\theta} - e^{-i\theta}$$

$$e^{i\theta} - 2ix - e^{-i\theta} = 0$$

$$e^{2i\theta} - 2ixe^{i\theta} - 1 = 0$$

$$e^{i\theta} = ix \pm \sqrt{-x^2 + 1} = ix \pm \sqrt{1 - x^2}$$

$$\begin{aligned} \therefore \sin^{-1} x = \theta &= \frac{1}{i} \log \left(ix \pm \sqrt{1 - x^2} \right) = \frac{1}{i} \log \left(\pm \sqrt{1 - x^2} + ix \right) \\ &= \frac{1}{i} \cdot i \arg \left(\pm \sqrt{1 - x^2} + ix \right) = \arg \left(\pm \sqrt{1 - x^2} + ix \right) \end{aligned}$$



付録: 積分③の補足

$$\begin{aligned}\frac{d}{dx} \left[\frac{1}{i} \log \left(ix \pm \sqrt{1-x^2} \right) \right] &= \frac{1}{i} \cdot \frac{i \mp \frac{x}{\sqrt{1-x^2}}}{ix \pm \sqrt{1-x^2}} \quad (\text{複号同順}) \\ &= \frac{1}{i\sqrt{1-x^2}} \cdot \frac{i\sqrt{1-x^2} \mp x}{ix \pm \sqrt{1-x^2}} \\ &= \frac{1}{\sqrt{1-x^2}} \cdot \frac{i\sqrt{1-x^2} \mp x}{-x \pm i\sqrt{1-x^2}} \\ &= \pm \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

付録：積分③の補足

したがって、

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \sin^{-1} x + C \quad (C \text{ は積分定数}) \\ &= \frac{1}{i} \log \left(ix + \sqrt{1-x^2} \right) + C\end{aligned}$$

付録: 積分④の検算

$$\begin{aligned}\frac{d}{dx} \left[\log \left(x + \sqrt{1 + x^2} \right) \right] &= \frac{1 + \frac{x}{\sqrt{1 + x^2}}}{x + \sqrt{1 + x^2}} \\ &= \frac{1}{\sqrt{1 + x^2}} \cdot \frac{\sqrt{1 + x^2} + x}{x + \sqrt{1 + x^2}} \\ &= \frac{1}{\sqrt{1 + x^2}}\end{aligned}$$

付録: 積分④の別解

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}} dx &= \frac{1}{i} \int \frac{1}{\sqrt{1-(ix)^2}} i dx \\&= \frac{1}{i} \cdot \frac{1}{i} \log \left(i \cdot ix + \sqrt{1-(ix)^2} \right) + C \quad (C \text{ は積分定数}) \\&= -\log \left(-x + \sqrt{1+x^2} \right) + C \\&= \log \frac{1}{-x + \sqrt{1+x^2}} + C \\&= \log \left(x + \sqrt{1+x^2} \right) + C\end{aligned}$$

付録：積分⑤の検算

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \text{ より}$$

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{2} \left(x\sqrt{1-x^2} + \sin^{-1} x \right) \right] &= \frac{1}{2} \left(\sqrt{1-x^2} - x \cdot \frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ &= \frac{1}{2} \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ &= \frac{1}{2} \left(\sqrt{1-x^2} + \frac{1-x^2}{\sqrt{1-x^2}} \right) \\ &= \frac{1}{2} \left(\sqrt{1-x^2} + \sqrt{1-x^2} \right) = \sqrt{1-x^2} \end{aligned}$$

付録：積分⑥の別解

$$\begin{aligned}\int \sqrt{1+x^2} dx &= \frac{1}{i} \int \sqrt{1-(ix)^2} i dx \\ &= \frac{1}{i} \cdot \frac{1}{2} \left\{ ix \sqrt{1-(ix)^2} + \frac{1}{i} \log \left(i \cdot ix + \sqrt{1-(ix)^2} \right) \right\} + C \\ &\quad (C \text{ は積分定数})\end{aligned}$$

$$\begin{aligned}&= \frac{1}{2} \left\{ x \sqrt{1+x^2} - \log \left(-x + \sqrt{1+x^2} \right) \right\} + C \\ &= \frac{1}{2} \left\{ x \sqrt{1+x^2} + \log \frac{1}{-x + \sqrt{1+x^2}} \right\} + C \\ &= \frac{1}{2} \left\{ x \sqrt{1+x^2} + \log \left(x + \sqrt{1+x^2} \right) \right\} + C\end{aligned}$$