$\sqrt{1+x^2}$ の積分
―5つの解法―

渡邉 俊夫

問題

不定積分

$$\int \sqrt{1+x^2}\mathrm{d}x$$

を求めなさい。

解①

$\cosh^2 t - \sinh^2 t = 1$ より
$1 + \sinh^2 t = \cosh^2 t$
を利用する

$x = \sinh t = \dfrac{e^t - e^{-t}}{2}$ とおくと

$$1 + x^2 = 1 + \left(\frac{e^t - e^{-t}}{2}\right)^2 = 1 + \frac{(e^t - e^{-t})^2}{4} = \frac{4 + (e^t - e^{-t})^2}{4}$$

$$= \frac{(e^t + e^{-t})^2}{4} = \left(\frac{e^t + e^{-t}}{2}\right)^2 = \cosh^2 t$$

$$dx = \frac{e^t + e^{-t}}{2} dt$$

より

解①

$$\int \sqrt{1+x^2}\mathrm{d}x = \int \frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\cdot\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\mathrm{d}t = \int \frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2}{4}\mathrm{d}t$$

$$= \int \frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}+2}{4}\mathrm{d}t = \int \left(\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}}{4}+\frac{1}{2}\right)\mathrm{d}t$$

$$= \frac{\mathrm{e}^{2t}-\mathrm{e}^{-2t}}{8}+\frac{1}{2}t+C$$ （C は積分定数）

$$= \frac{(\mathrm{e}^t-\mathrm{e}^{-t})(\mathrm{e}^t+\mathrm{e}^{-t})}{8}+\frac{1}{2}t+C$$

$$= \frac{1}{2}\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}\cdot\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}+\frac{1}{2}t+C$$

解①

ここで、$x = \dfrac{\mathrm{e}^t - \mathrm{e}^{-t}}{2}$ より

$$\mathrm{e}^t - \mathrm{e}^{-t} = 2x$$

$$\mathrm{e}^{2t} - 1 = 2x\mathrm{e}^t$$

$$\mathrm{e}^{2t} - 2x\mathrm{e}^t - 1 = 0$$

これを解くと、$\mathrm{e}^t = x \pm \sqrt{x^2 + 1}$ となるが、$\mathrm{e}^t > 0$ だから

$$\mathrm{e}^t = x + \sqrt{x^2 + 1}$$

$$\therefore t = \log\left(x + \sqrt{x^2 + 1}\right) = \log\left(x + \sqrt{1 + x^2}\right)$$

解①

以上より

$$\int\sqrt{1+x^2}\mathrm{d}x=\int\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\cdot\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\mathrm{d}t$$

$$=\frac{1}{2}\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}\cdot\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}+\frac{1}{2}t+C$$

$$=\frac{1}{2}x\cdot\sqrt{1+x^2}+\frac{1}{2}\log\left(x+\sqrt{1+x^2}\right)+C$$

$$=\frac{1}{2}\left\{x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)\right\}+C$$

検算

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\left\{x\sqrt{1+x^2}+\log\left(x+\sqrt{1+x^2}\right)\right\}\right]$$

$$=\frac{1}{2}\left\{\sqrt{1+x^2}+x\frac{x}{\sqrt{1+x^2}}+\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\right\}$$

$$=\frac{1}{2}\left\{\sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\frac{\sqrt{1+x^2}+x}{x+\sqrt{1+x^2}}\right\}$$

$$=\frac{1}{2}\left\{\sqrt{1+x^2}+\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right\}$$

検算

$$=\frac{1}{2}\left\{\sqrt{1+x^2}+\frac{x^2+1}{\sqrt{1+x^2}}\right\}=\frac{1}{2}\left\{\sqrt{1+x^2}+\frac{1+x^2}{\sqrt{1+x^2}}\right\}$$

$$=\frac{1}{2}\left\{\sqrt{1+x^2}+\sqrt{1+x^2}\right\}$$

$$=\sqrt{1+x^2}$$

解②

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \int (x)' \cdot \sqrt{1+x^2}\,\mathrm{d}x = x\sqrt{1+x^2} - \int x \cdot \left(\sqrt{1+x^2}\right)' \mathrm{d}x$$

$$= x\sqrt{1+x^2} - \int x \cdot \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = x\sqrt{1+x^2} - \int \frac{1+x^2-1}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$= x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,\mathrm{d}x + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$$

より

$$2\int \sqrt{1+x^2}\,\mathrm{d}x = x\sqrt{1+x^2} + \int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$$

解②

ここで、$x=\sinh t=\dfrac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}$ とおくと

$$1+x^2=1+\left(\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}\right)^2=1+\frac{(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}=\frac{4+(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$$

$$=\frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2}{4}=\left(\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\right)^2=\cosh^2 t$$

$$\mathrm{d}x=\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}\mathrm{d}t=\cosh t\,\mathrm{d}t$$

より

解②

$$\int \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \int \frac{1}{\cosh t}\cosh t\,\mathrm{d}t = \int \mathrm{d}t$$

$$= t + C' \qquad （C' \text{ は積分定数}）$$

$$= \sinh^{-1} x + C'$$

解②

ここで、$x=\sinh t=\dfrac{\mathrm{e}^{t}-\mathrm{e}^{-t}}{2}$ より

$$\mathrm{e}^{t}-\mathrm{e}^{-t}=2x$$

$$\mathrm{e}^{2t}-1=2x\mathrm{e}^{t}$$

$$\mathrm{e}^{2t}-2x\mathrm{e}^{t}-1=0$$

これを解くと、$\mathrm{e}^{t}=x\pm\sqrt{x^2+1}$ となるが、$\mathrm{e}^{t}>0$ だから

$$\mathrm{e}^{t}=x+\sqrt{x^2+1}$$

$$\therefore t=\log\left(x+\sqrt{x^2+1}\right)=\log\left(x+\sqrt{1+x^2}\right)$$

解②

以上より

$$\int \sqrt{1+x^2}\mathrm{d}x = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$$

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1}x + C$$ （C は積分定数）

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\log\left(x+\sqrt{1+x^2}\right) + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x+\sqrt{1+x^2}\right)\right\} + C$$

解③

$x + \sqrt{1 + x^2} = u$ とおくと

置換 $\sqrt{1 + x^2} = u - x$ を
オイラー置換という

$$\sqrt{1 + x^2} = u - x$$

$$1 + x^2 = (u - x)^2 = u^2 - 2ux + x^2$$

$$1 = u^2 - 2ux$$

$$2ux = u^2 - 1$$

$$\therefore x = \frac{1}{2u}(u^2 - 1) = \frac{1}{2}\left(u - \frac{1}{u}\right)$$

より

解③

$$\mathrm{d}x = \frac{1}{2}\left(1+\frac{1}{u^2}\right)\mathrm{d}u$$

だから

$$\int\sqrt{1+x^2}\mathrm{d}x = \int\sqrt{1+\frac{1}{4}\left(u-\frac{1}{u}\right)^2}\cdot\frac{1}{2}\left(1+\frac{1}{u^2}\right)\mathrm{d}u$$

$$= \int\frac{1}{2}\sqrt{4+\left(u-\frac{1}{u}\right)^2}\cdot\frac{1}{2}\left(1+\frac{1}{u^2}\right)\mathrm{d}t = \int\frac{1}{2}\sqrt{\left(u+\frac{1}{u}\right)^2}\cdot\frac{1}{2}\left(1+\frac{1}{u^2}\right)\mathrm{d}u$$

$$= \frac{1}{4}\int\left(u+\frac{1}{u}\right)\left(1+\frac{1}{u^2}\right)\mathrm{d}t = \frac{1}{4}\int\left(u+\frac{1}{u^3}+\frac{2}{u}\right)\mathrm{d}t$$

解③

$$= \frac{1}{4}\left(\frac{u^2}{2} - \frac{1}{2u^2} + 2\log u\right) + C \quad (C\text{ は積分定数})$$

$$= \frac{1}{8}\left(u^2 - \frac{1}{u^2}\right) + \frac{1}{2}\log u + C = \frac{1}{8}\left(u - \frac{1}{u}\right)\left(u + \frac{1}{u}\right) + \frac{1}{2}\log u + C$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(u - \frac{1}{u}\right)\cdot\frac{1}{2}\left(u + \frac{1}{u}\right) + \frac{1}{2}\log u + C$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(u - \frac{1}{u}\right)\cdot\left(u - \frac{1}{2}\left(u - \frac{1}{u}\right)\right) + \frac{1}{2}\log u + C$$

$$= \frac{1}{2}\cdot x\cdot(u - x) + \frac{1}{2}\log u + C$$

解③

$$= \frac{1}{2} \cdot x \cdot \sqrt{1+x^2} + \frac{1}{2} \log\left(x + \sqrt{1+x^2}\right) + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x + \sqrt{1+x^2}\right)\right\} + C$$

解④

$\cos^2\theta + \sin^2\theta = 1$ より

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

を利用する

$x = \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ とおくと

$$1 + x^2 = 1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\mathrm{d}x = \frac{\cos\theta\cdot\cos\theta - \sin\theta\cdot(-\sin\theta)}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{\cos^2\theta}\mathrm{d}\theta$$

より

$$\int\sqrt{1+x^2}\mathrm{d}x = \int\frac{1}{\cos\theta}\frac{1}{\cos^2\theta}\mathrm{d}\theta = \int\frac{1}{\cos^3\theta}\mathrm{d}\theta$$

$$= \int\frac{1}{\cos\theta}(\tan\theta)'\mathrm{d}\theta = \frac{1}{\cos\theta}\tan\theta - \int\left(\frac{1}{\cos\theta}\right)'\tan\theta\,\mathrm{d}\theta$$

解④

$$= \frac{\tan\theta}{\cos\theta} - \int \frac{\sin\theta}{\cos^2\theta}\tan\theta\,\mathrm{d}\theta = \frac{\tan\theta}{\cos\theta} - \int \frac{\sin^2\theta}{\cos^3\theta}\mathrm{d}\theta$$

$$= \frac{\tan\theta}{\cos\theta} - \int \frac{1-\cos^2\theta}{\cos^3\theta}\mathrm{d}\theta$$

$$= \frac{\tan\theta}{\cos\theta} - \int \frac{1}{\cos^3\theta}\mathrm{d}\theta + \int \frac{1}{\cos\theta}\mathrm{d}\theta$$

したがって、

$$\int \sqrt{1+x^2}\mathrm{d}x = \int \frac{1}{\cos^3\theta}\mathrm{d}\theta = \frac{1}{2}\left\{\frac{\tan\theta}{\cos\theta} + \int \frac{1}{\cos\theta}\mathrm{d}\theta\right\}$$

解④

ここで、

$$\int \frac{1}{\cos\theta}\,\mathrm{d}\theta = \int \frac{\cos\theta}{\cos^2\theta}\,\mathrm{d}\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\,\mathrm{d}\theta$$

$$= \int \frac{\cos\theta}{(1+\sin\theta)(1-\sin\theta)}\,\mathrm{d}\theta = \frac{1}{2}\int \left(\frac{\cos\theta}{1+\sin\theta} + \frac{\cos\theta}{1-\sin\theta}\right)\mathrm{d}\theta$$

$$= \frac{1}{2}\left(\log(1+\sin\theta) - \log(1-\sin\theta)\right) + C' \qquad (C' \text{ は積分定数})$$

$$= \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} + C' = \frac{1}{2}\log\frac{(1+\sin\theta)^2}{1-\sin^2\theta} + C'$$

解④

$$= \frac{1}{2}\log\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 + C' = \log\left(\frac{1+\sin\theta}{\cos\theta}\right) + C'$$

$$= \log\left(\frac{1}{\cos\theta} + \tan\theta\right) + C' = \log\left(\tan\theta + \frac{1}{\cos\theta}\right) + C'$$

であるから

$$\int\sqrt{1+x^2}\mathrm{d}x = \frac{1}{2}\left\{\frac{\tan\theta}{\cos\theta} + \log\left(\tan\theta + \frac{1}{\cos\theta}\right)\right\} + C \quad (C\text{ は積分定数})$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x+\sqrt{1+x^2}\right)\right\} + C$$

解⑤

$\cos^2\theta + \sin^2\theta = 1$ より

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

を利用する

$$x = \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ とおくと}$$

$$1 + x^2 = 1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\mathrm{d}x = \frac{\cos\theta \cdot \cos\theta - \sin\theta \cdot (-\sin\theta)}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{\cos^2\theta}\mathrm{d}\theta$$

より

$$\int \sqrt{1 + x^2}\mathrm{d}x = \int \frac{1}{\cos\theta}\frac{1}{\cos^2\theta}\mathrm{d}\theta = \int \frac{1}{\cos^3\theta}\mathrm{d}\theta$$

$$= \int \frac{\cos\theta}{\cos^4\theta}\mathrm{d}\theta = \int \frac{\cos\theta}{(1 - \sin^2\theta)^2}\mathrm{d}\theta$$

解⑤

$$= \int \frac{\cos\theta}{(1+\sin\theta)^2(1-\sin\theta)^2}\,\mathrm{d}\theta$$

$$= \int \frac{1}{2}\left\{\frac{1}{2}\left(\frac{1}{(1+\sin\theta)^2}+\frac{1}{(1-\sin\theta)^2}\right)+\frac{1}{(1+\sin\theta)(1-\sin\theta)}\right\}\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{2}\left\{\frac{1}{2}\left(\frac{1}{(1+\sin\theta)^2}+\frac{1}{(1-\sin\theta)^2}\right)\right.$$

$$\left.+\frac{1}{2}\left(\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}\right)\right\}\cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \left\{\frac{1}{(1+\sin\theta)^2}+\frac{1}{(1-\sin\theta)^2}+\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}\right\}\cos\theta\,\mathrm{d}\theta$$

解⑤

$$= \frac{1}{4}\left\{-\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}+\log(1+\sin\theta)-\log(1-\sin\theta)\right\}+C$$

（C は積分定数）

$$= \frac{1}{4}\left\{\frac{2\sin\theta}{(1+\sin\theta)(1-\sin\theta)}+\log\frac{1+\sin\theta}{1-\sin\theta}\right\}+C$$

$$= \frac{1}{4}\left\{\frac{2\sin\theta}{1-\sin^2\theta}+\log\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right\}+C$$

$$= \frac{1}{4}\left\{\frac{2\sin\theta}{\cos^2\theta}+\log\frac{(1+\sin\theta)^2}{\cos^2\theta}\right\}+C$$

解⑤

$$= \frac{1}{2}\left\{\frac{\sin\theta}{\cos^2\theta} + \log\frac{1+\sin\theta}{\cos\theta}\right\} + C = \frac{1}{2}\left\{\frac{\tan\theta}{\cos\theta} + \log\left(\frac{1}{\cos\theta} + \tan\theta\right)\right\} + C$$

$$= \frac{1}{2}\left\{\frac{\tan\theta}{\cos\theta} + \log\left(\tan\theta + \frac{1}{\cos\theta}\right)\right\} + C$$

$$= \frac{1}{2}\left\{x\sqrt{1+x^2} + \log\left(x+\sqrt{1+x^2}\right)\right\} + C$$