4 次方程式の判別式

実数を係数とする 4 次方程式

$$ax^4+bx^3+cx^2+dx+e=0$$

の 4 つの解を $x=\alpha,\beta,\gamma,\delta$ とすると、その判別式は

$$D=a^6(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\alpha)^2(\alpha-\gamma)^2(\beta-\delta)^2$$

で定義される。

この方程式は、$x=\alpha,\beta,\gamma,\delta$ が解であることより

$$a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$$

の形で表せるはずである。左辺を展開すると

$$\begin{aligned}&a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\\&=a(x^2-(\alpha+\beta)x+\alpha\beta)(x^2-(\gamma+\delta)x+\gamma\delta)\\&=a(x^4-(\alpha+\beta+\gamma+\delta)x^3+(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)x^2-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)x\\&\qquad+\alpha\beta\gamma\delta)\end{aligned}$$

となるから、元の方程式と各項を比較して、解と係数の関係

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$$

$$\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta=\frac{c}{a}$$

$$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=-\frac{d}{a}$$

$$\alpha\beta\gamma\delta=\frac{e}{a}$$

を得る。

したがって、判別式 D を、$\alpha,\ \beta,\ \gamma,\ \delta$ の基本対称式

$$e_1=\alpha+\beta+\gamma+\delta$$

$$e_2=\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta$$

$$e_3=\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta$$

$$e_4=\alpha\beta\gamma\delta$$

によって表せば、判別式の係数表示が得られる。

まず、

$$\begin{aligned}&(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)=-\beta\gamma\delta+(\beta\gamma+\gamma\delta+\delta\beta)\alpha-(\beta+\gamma+\delta)\alpha^2+\alpha^3\\&=-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+(\alpha\beta\gamma+\gamma\delta\alpha+\delta\alpha\beta)+(\beta\gamma+\gamma\delta+\delta\beta)\alpha-(\beta+\gamma+\delta)\alpha^2+\alpha^3\\&=-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+(\beta\gamma+\gamma\delta+\delta\beta)\alpha+(\beta\gamma+\gamma\delta+\delta\beta)\alpha-(\beta+\gamma+\delta)\alpha^2+\alpha^3\\&=-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+2(\beta\gamma+\gamma\delta+\delta\beta)\alpha-(\beta+\gamma+\delta)\alpha^2+\alpha^3\\&=-(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha-2(\alpha\beta+\delta\alpha+\alpha\gamma)\alpha\\&\qquad-(\beta+\gamma+\delta)\alpha^2+\alpha^3\end{aligned}$$

$$= -(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) + 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha - 2(\beta + \delta + \alpha)\alpha^2$$
$$- (\beta + \gamma + \delta)\alpha^2 + \alpha^3$$
$$= -(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) + 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha - 3(\beta + \gamma + \delta)\alpha^2 + \alpha^3$$
$$= -(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) + 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha - 3(\alpha + \beta + \gamma + \delta)\alpha^2$$
$$+ 3\alpha^3 + \alpha^3$$
$$= -(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) + 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha - 3(\alpha + \beta + \gamma + \delta)\alpha^2 + 4\alpha^3$$

より

$$(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta) = -e_3 + 2e_2\alpha - 3e_1\alpha^2 + 4\alpha^3$$

となる。同様に、

$$(\beta - \alpha)(\beta - \gamma)(\beta - \delta) = -e_3 + 2e_2\beta - 3e_1\beta^2 + 4\beta^3$$
$$(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta) = -e_3 + 2e_2\gamma - 3e_1\gamma^2 + 4\gamma^3$$
$$(\delta - \alpha)(\delta - \beta)(\delta - \gamma) = -e_3 + 2e_2\delta - 3e_1\delta^2 + 4\delta^3$$

である。

簡単のため、$b = 0$ のとき、すなわち、方程式が

$$ax^4 + cx^2 + dx + e = 0$$

の場合を考える。このとき、

$$e_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a} = 0$$

であるから

$$(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta) = -e_3 + 2e_2\alpha + 4\alpha^3$$
$$(\beta - \alpha)(\beta - \gamma)(\beta - \delta) = -e_3 + 2e_2\beta + 4\beta^3$$
$$(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta) = -e_3 + 2e_2\gamma + 4\gamma^3$$
$$(\delta - \alpha)(\delta - \beta)(\delta - \gamma) = -e_3 + 2e_2\delta + 4\delta^3$$

となる。したがって、

$$D/a^6 = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \delta)^2(\delta - \alpha)^2(\alpha - \gamma)^2(\beta - \delta)^2$$
$$= (-e_3 + 2e_2\alpha + 4\alpha^3)(-e_3 + 2e_2\beta + 4\beta^3)(-e_3 + 2e_2\gamma + 4\gamma^3)(e_3 + 2e_2\delta + 4\delta^3)$$
$$= {e_3}^4 - 2e_2{e_3}^3(\alpha + \beta + \gamma + \delta) - 4{e_3}^3(\alpha^3 + \beta^3 + \gamma^3 + \delta^3)$$
$$+ 4{e_2}^2{e_3}^2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 8e_2{e_3}^2\big(\alpha^3(\beta + \gamma + \delta) + \beta^3(\alpha + \gamma + \delta) + \gamma^3(\alpha + \beta + \delta) + \delta^3(\alpha + \beta + \gamma)\big)$$
$$+ 16{e_3}^2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\delta^3 + \delta^3\alpha^3 + \alpha^3\gamma^3 + \delta^3\beta^3)$$
$$- 8{e_2}^3e_3(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)$$
$$- 16{e_2}^2e_3\big(\alpha^3(\beta\gamma + \gamma\delta + \delta\beta) + \beta^3(\gamma\delta + \delta\alpha + \alpha\gamma) + \gamma^3(\alpha\beta + \delta\alpha + \delta\beta) + \delta^3(\alpha\beta + \beta\gamma + \alpha\gamma)\big)$$
$$- 32e_2e_3\big(\alpha^3\beta^3(\gamma + \delta) + \beta^3\gamma^3(\delta + \alpha) + \gamma^3\delta^3(\alpha + \beta) + \delta^3\alpha^3(\beta + \gamma) + \alpha^3\gamma^3(\beta + \delta)$$
$$+ \delta^3\beta^3(\alpha + \gamma)\big)$$

$$- 64e_3(\alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \gamma^3\delta^3\alpha^3 + \delta^3\alpha^3\beta^3)$$
$$+ 16{e_2}^4\alpha\beta\gamma\delta$$
$$+ 32{e_2}^3(\alpha\beta\gamma\delta^3 + \beta\gamma\delta\alpha^3 + \gamma\delta\alpha\beta^3 + \delta\alpha\beta\gamma^3)$$
$$+ 64{e_2}^2(\alpha\beta\gamma^3\delta^3 + \beta\gamma\delta^3\alpha^3 + \gamma\delta\alpha^3\beta^3 + \delta\alpha\beta^3\gamma^3 + \alpha\gamma\delta^3\beta^3 + \delta\beta\alpha^3\gamma^3)$$
$$+ 128e_2(\alpha\beta^3\gamma^3\delta^3 + \beta\alpha^3\gamma^3\delta^3 + \gamma\alpha^3\beta^3\delta^3 + \delta\alpha^3\beta^3\gamma^3)$$
$$+ 256\alpha^3\beta^3\gamma^3\delta^3$$

$$= {e_3}^4 - 0 - 4{e_3}^3(\alpha^3 + \beta^3 + \gamma^3 + \delta^3)$$
$$+ 4{e_2}^3{e_3}^2$$
$$+ 8e_2{e_3}^2\big(\alpha^3(\beta + \gamma + \delta) + \beta^3(\alpha + \gamma + \delta) + \gamma^3(\alpha + \beta + \delta) + \delta^3(\alpha + \beta + \gamma)\big)$$
$$+ 16{e_3}^2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\delta^3 + \delta^3\alpha^3 + \alpha^3\gamma^3 + \delta^3\beta^3)$$
$$- 8{e_2}^3{e_3}^2$$
$$- 16{e_2}^2e_3\big(\alpha^3(\beta\gamma + \gamma\delta + \delta\beta) + \beta^3(\gamma\delta + \delta\alpha + \alpha\gamma) + \gamma^3(\alpha\beta + \delta\alpha + \delta\beta) + \delta^3(\alpha\beta + \beta\gamma + \alpha\gamma)\big)$$
$$- 32e_2e_3\big(\alpha^3\beta^3(\gamma + \delta) + \beta^3\gamma^3(\delta + \alpha) + \gamma^3\delta^3(\alpha + \beta) + \delta^3\alpha^3(\beta + \gamma) + \alpha^3\gamma^3(\beta + \delta) + \delta^3\beta^3(\alpha + \gamma)\big)$$
$$- 64e_3(\alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \gamma^3\delta^3\alpha^3 + \delta^3\alpha^3\beta^3)$$
$$+ 16{e_2}^4e_4$$
$$+ 32{e_2}^3(\delta^2 + \alpha^2 + \beta^2 + \gamma^2)\,\alpha\beta\gamma\delta$$
$$+ 64{e_2}^2(\gamma^2\delta^2 + \delta^2\alpha^2 + \alpha^2\beta^2 + \beta^2\gamma^2 + \delta^2\beta^2 + \alpha^2\gamma^2)\alpha\beta\gamma\delta$$
$$+ 128e_2(\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2)\alpha\beta\gamma\delta$$
$$+ 256{e_4}^3$$

$$= {e_3}^4 - 4{e_3}^3(\alpha^3 + \beta^3 + \gamma^3 + \delta^3)$$
$$+ 4{e_2}^3{e_3}^2$$
$$+ 8e_2{e_3}^2\big(\alpha^3(\beta + \gamma + \delta) + \beta^3(\alpha + \gamma + \delta) + \gamma^3(\alpha + \beta + \delta) + \delta^3(\alpha + \beta + \gamma)\big)$$
$$+ 16{e_3}^2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\delta^3 + \delta^3\alpha^3 + \alpha^3\gamma^3 + \delta^3\beta^3)$$
$$- 8{e_2}^3{e_3}^2$$
$$- 16{e_2}^2e_3\big(\alpha^3(\beta\gamma + \gamma\delta + \delta\beta) + \beta^3(\gamma\delta + \delta\alpha + \alpha\gamma) + \gamma^3(\alpha\beta + \delta\alpha + \delta\beta) + \delta^3(\alpha\beta + \beta\gamma + \alpha\gamma)\big)$$
$$- 32e_2e_3\big(\alpha^3\beta^3(\gamma + \delta) + \beta^3\gamma^3(\delta + \alpha) + \gamma^3\delta^3(\alpha + \beta) + \delta^3\alpha^3(\beta + \gamma) + \alpha^3\gamma^3(\beta + \delta) + \delta^3\beta^3(\alpha + \gamma)\big)$$
$$- 64e_3(\alpha^3\beta^3\gamma^3 + \beta^3\gamma^3\delta^3 + \gamma^3\delta^3\alpha^3 + \delta^3\alpha^3\beta^3)$$
$$+ 16{e_2}^4e_4$$
$$+ 32{e_2}^3e_4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$$
$$+ 64{e_2}^2e_4(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\delta^2 + \delta^2\alpha^2 + \alpha^2\gamma^2 + \delta^2\beta^2)$$
$$+ 128e_2e_4(\alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \gamma^2\delta^2\alpha^2 + \delta^2\alpha^2\beta^2)$$
$$+ 256{e_4}^3$$

となる。

続いて、判別式を展開したときに現れる対称式を基本対称式で表す。
先に、次数の低いものから計算すると

$$
\begin{aligned}
\alpha^2+\beta^2+\gamma^2+\delta^2 &= (\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\\
&= {e_1}^2-2e_2\\
&= -2e_2
\end{aligned}
$$

$$
\begin{aligned}
&\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2+\alpha^2\gamma^2+\delta^2\beta^2\\
&= (\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2\\
&\quad -2\alpha(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad -2\beta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad -2\gamma(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad -2\delta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad +2\alpha\beta\gamma\delta\\
&= (\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2-2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+2\alpha\beta\gamma\delta\\
&= {e_2}^2-2e_1e_3+2e_4\\
&= {e_2}^2+2e_4
\end{aligned}
$$

$$
\begin{aligned}
&\alpha^2\beta^2\gamma^2+\beta^2\gamma^2\delta^2+\gamma^2\delta^2\alpha^2+\delta^2\alpha^2\beta^2\\
&= (\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha\beta\gamma\delta\\
&= {e_3}^2-2e_2e_4
\end{aligned}
$$

である。これらを用いて

$$
\begin{aligned}
&\alpha^3+\beta^3+\gamma^3+\delta^3\\
&= (\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2)\\
&\quad -\alpha^2(\beta+\gamma+\delta)-\beta^2(\alpha+\gamma+\delta)-\gamma^2(\alpha+\beta+\delta)-\delta^2(\alpha+\beta+\gamma)\\
&= (\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2)\\
&\quad -\alpha(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)+(\alpha\beta\gamma+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad -\beta(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)+(\alpha\beta\gamma+\beta\gamma\delta+\delta\alpha\beta)\\
&\quad -\gamma(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)+(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha)\\
&\quad -\delta(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)+(\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&= (\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2)\\
&\quad -(\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\\
&\quad +3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&= (\alpha+\beta+\gamma+\delta)\big((\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\big)
\end{aligned}
$$

$$-(\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$+3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)$$
$$=(\alpha+\beta+\gamma+\delta)^3-3(\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$+3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)$$
$$={e_1}^3-3e_1e_2+3e_3$$
$$=3e_3$$

$$\alpha^3(\beta+\gamma+\delta)+\beta^3(\alpha+\gamma+\delta)+\gamma^3(\alpha+\beta+\delta)+\delta^3(\alpha+\beta+\gamma)$$
$$=\alpha^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\alpha^2(\beta\gamma+\gamma\delta+\delta\beta)$$
$$+\beta^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\beta^2(\gamma\delta+\delta\alpha+\alpha\gamma)$$
$$+\gamma^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\gamma^2(\alpha\beta+\delta\alpha+\delta\beta)$$
$$+\delta^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\delta^2(\alpha\beta+\beta\gamma+\alpha\gamma)$$
$$=\alpha^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\alpha(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+\alpha\beta\gamma\delta$$
$$+\beta^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\beta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+\alpha\beta\gamma\delta$$
$$+\gamma^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\gamma(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+\alpha\beta\gamma\delta$$
$$+\delta^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-\delta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+\alpha\beta\gamma\delta$$
$$=(\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$-(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+4\alpha\beta\gamma\delta$$
$$=\left((\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\right)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$-(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+4\alpha\beta\gamma\delta$$
$$=(\alpha+\beta+\gamma+\delta)^2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2$$
$$-(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+4\alpha\beta\gamma\delta$$
$$={e_1}^2e_2-2{e_2}^2-e_1e_3+4e_4$$
$$=-2{e_2}^2+4e_4$$

$$\alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\delta^3+\delta^3\alpha^3+\alpha^3\gamma^3+\delta^3\beta^3$$
$$=(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2+\alpha^2\gamma^2+\delta^2\beta^2)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$-(\alpha^2\beta\gamma+\alpha^2\gamma\delta+\alpha^2\delta\beta+\beta^2\alpha\gamma+\beta^2\gamma\delta+\beta^2\delta\alpha+\gamma^2\alpha\beta+\gamma^2\beta\delta+\gamma^2\delta\alpha+\delta^2\alpha\beta+\delta^2\beta\gamma$$
$$+\delta^2\gamma\alpha+3\alpha\beta\gamma\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$+3(\alpha^2\beta^2\gamma\delta+\alpha\beta^2\gamma^2\delta+\alpha\beta\gamma^2\delta^2+\alpha^2\beta\gamma\delta^2+\alpha^2\beta\gamma^2\delta+\alpha\beta^2\gamma\delta^2+\alpha^3\beta\gamma\delta+\alpha\beta^3\gamma\delta+\alpha\beta\gamma^3\delta$$
$$+\alpha\beta\gamma\delta^3+\alpha^2\beta^2\gamma^2+\beta^2\gamma^2\delta^2+\gamma^2\delta^2\alpha^2+\delta^2\alpha^2\beta^2)$$
$$=((\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2-2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)+2\alpha\beta\gamma\delta)(\alpha\beta$$
$$+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$-\left((\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)(\alpha+\beta+\gamma+\delta)-4\alpha\beta\gamma\delta\right)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)$$
$$+3(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha\beta\gamma\delta$$
$$+3(\alpha^2+\beta^2+\gamma^2+\delta^2)\alpha\beta\gamma\delta$$

$$+ 3(\alpha^2\beta^2\gamma^2 + \beta^2\gamma^2\delta^2 + \gamma^2\delta^2\alpha^2 + \delta^2\alpha^2\beta^2)$$
$$= (\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)^3$$
$$- 2(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 2\alpha\beta\gamma\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$- (\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha + \beta + \gamma + \delta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 4\alpha\beta\gamma\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 3(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha\beta\gamma\delta$$
$$+ 3\big((\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\big)\alpha\beta\gamma\delta$$
$$+ 3((\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha\beta\gamma\delta)$$
$$= (\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)^3$$
$$- 3(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 9\alpha\beta\gamma\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 3(\alpha + \beta + \gamma + \delta)^2\alpha\beta\gamma\delta - 6(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha\beta\gamma\delta$$
$$+ 3(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^2 - 6(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\alpha\beta\gamma\delta$$
$$= (\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)^3$$
$$- 3(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$- 3\alpha\beta\gamma\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)$$
$$+ 3(\alpha + \beta + \gamma + \delta)^2\alpha\beta\gamma\delta$$
$$+ 3(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^2$$
$$= {e_2}^3 - 3e_1e_2e_3 - 3e_2e_4 + 3{e_1}^2e_4 + 3{e_3}^2$$
$$= {e_2}^3 - 3e_2e_4 + 3{e_3}^2$$

$$\alpha^3(\beta\gamma + \gamma\delta + \delta\beta) + \beta^3(\gamma\delta + \delta\alpha + \alpha\gamma) + \gamma^3(\alpha\beta + \delta\alpha + \delta\beta) + \delta^3(\alpha\beta + \beta\gamma + \alpha\gamma)$$
$$= \alpha^2(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - \alpha^2\beta\gamma\delta$$
$$+ \beta^2(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - \alpha\beta^2\gamma\delta$$
$$+ \gamma^2(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - \alpha\beta\gamma^2\delta$$
$$+ \delta^2(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - \alpha\beta\gamma\delta^2$$
$$= (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - (\alpha + \beta + \gamma + \delta)\alpha\beta\gamma\delta$$
$$= \big((\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\big)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)$$
$$- (\alpha + \beta + \gamma + \delta)\alpha\beta\gamma\delta$$
$$= (\alpha + \beta + \gamma + \delta)^2(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)$$
$$- 2(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)$$
$$- (\alpha + \beta + \gamma + \delta)\alpha\beta\gamma\delta$$
$$= {e_1}^2e_3 - 2e_2e_3 - e_1e_4$$

$= -2e_2e_3$

$$\alpha^3\beta^3(\gamma+\delta)+\beta^3\gamma^3(\delta+\alpha)+\gamma^3\delta^3(\alpha+\beta)+\delta^3\alpha^3(\beta+\gamma)+\alpha^3\gamma^3(\beta+\delta)+\delta^3\beta^3(\alpha+\gamma)$$

$$\begin{aligned}
&= \alpha^2\beta^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha^2\beta^3\gamma\delta+\alpha^3\beta^2\gamma\delta)\\
&\quad+\beta^2\gamma^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha\beta^3\gamma^2\delta+\alpha\beta^2\gamma^3\delta)\\
&\quad+\gamma^2\delta^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha\beta\gamma^3\delta^2+\alpha\beta\gamma^2\delta^3)\\
&\quad+\delta^2\alpha^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha^2\beta\gamma\delta^3+\alpha^3\beta\gamma\delta^2)\\
&\quad+\alpha^2\gamma^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha^3\beta\gamma^2\delta+\alpha^2\beta\gamma^3\delta)\\
&\quad+\delta^2\beta^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-(\alpha\beta^2\gamma\delta^3+\alpha\beta^3\gamma\delta^2)\\
&= (\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2+\alpha^2\gamma^2+\delta^2\beta^2)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-(\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma^2\delta+\gamma\delta^2+\delta^2\alpha+\delta\alpha^2+\alpha^2\gamma+\alpha\gamma^2+\delta^2\beta+\delta\beta^2)\alpha\beta\gamma\delta\\
&= ((\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2-2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\qquad\qquad+8\alpha\beta\gamma\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-\big((\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)-3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\big)\alpha\beta\gamma\delta\\
&= (\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)^2\\
&\quad+8\alpha\beta\gamma\delta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-(\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha\beta\gamma\delta\\
&\quad-3(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\alpha\beta\gamma\delta\\
&= (\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-2(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)^2\\
&\quad+5\alpha\beta\gamma\delta(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-(\alpha+\beta+\gamma+\delta)(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha\beta\gamma\delta\\
&= {e_2}^2e_3-2e_1{e_3}^2+5e_3e_4-e_1e_2e_4\\
&= {e_2}^2e_3+5e_3e_4
\end{aligned}$$

$$\alpha^3\beta^3\gamma^3+\beta^3\gamma^3\delta^3+\gamma^3\delta^3\alpha^3+\delta^3\alpha^3\beta^3$$

$$\begin{aligned}
&= (\alpha^2\beta^2\gamma^2+\beta^2\gamma^2\delta^2+\gamma^2\delta^2\alpha^2+\delta^2\alpha^2\beta^2)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-\alpha^2\beta^2\gamma^2(\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-\beta^2\gamma^2\delta^2(\alpha\beta\gamma+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-\gamma^2\delta^2\alpha^2(\alpha\beta\gamma+\beta\gamma\delta+\delta\alpha\beta)\\
&\quad-\delta^2\alpha^2\beta^2(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha)\\
&= ((\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)^2-2(\alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha+\alpha\gamma+\delta\beta)\alpha\beta\gamma\delta)\\
&\qquad\cdot(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)\\
&\quad-\alpha^2\beta^2\gamma^2\delta(\beta\gamma+\gamma\alpha+\alpha\beta)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \beta^2\gamma^2\delta^2\alpha(\beta\gamma + \gamma\delta + \delta\beta)\\
&\quad - \gamma^2\delta^2\alpha^2\beta(\alpha\gamma + \gamma\delta + \delta\alpha)\\
&\quad - \delta^2\alpha^2\beta^2\gamma(\alpha\beta + \beta\delta + \delta\alpha)\\
&= (\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^3\\
&\quad - 2\alpha\beta\gamma\delta(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\\
&\quad - \alpha^2\beta^2\gamma^2\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \alpha^2\beta^2\gamma^2\delta(\gamma\delta + \delta\alpha + \delta\beta)\\
&\quad - \beta^2\gamma^2\delta^2\alpha(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \beta^2\gamma^2\delta^2\alpha(\alpha\beta + \delta\alpha + \alpha\gamma)\\
&\quad - \gamma^2\delta^2\alpha^2\beta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \gamma^2\delta^2\alpha^2\beta(\alpha\beta + \beta\gamma + \delta\beta)\\
&\quad - \delta^2\alpha^2\beta^2\gamma(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \delta^2\alpha^2\beta^2\gamma(\beta\gamma + \gamma\delta + \alpha\gamma)\\
&= (\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^3\\
&\quad - 2\alpha\beta\gamma\delta(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\\
&\quad - \alpha^2\beta^2\gamma^2\delta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \alpha^2\beta^2\gamma^2\delta^2(\gamma + \alpha + \beta)\\
&\quad - \beta^2\gamma^2\delta^2\alpha(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \beta^2\gamma^2\delta^2\alpha^2(\beta + \delta + \gamma)\\
&\quad - \gamma^2\delta^2\alpha^2\beta(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \gamma^2\delta^2\alpha^2\beta^2(\alpha + \gamma + \delta)\\
&\quad - \delta^2\alpha^2\beta^2\gamma(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta) + \delta^2\alpha^2\beta^2\gamma^2(\beta + \delta + \alpha)\\
&= (\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^3\\
&\quad - 2\alpha\beta\gamma\delta(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\\
&\quad - \alpha\beta\gamma\delta(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\\
&\quad + 3\alpha^2\beta^2\gamma^2\delta^2(\alpha + \beta + \gamma + \delta)\\
&= (\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)^3\\
&\quad - 3\alpha\beta\gamma\delta(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)(\alpha\beta + \beta\gamma + \gamma\delta + \delta\alpha + \alpha\gamma + \delta\beta)\\
&\quad + 3\alpha^2\beta^2\gamma^2\delta^2(\alpha + \beta + \gamma + \delta)\\
&= {e_3}^3 - 3e_2e_3e_4 + 3e_1{e_4}^2\\
&= {e_3}^3 - 3e_2e_3e_4
\end{aligned}
$$

となる。

以上より、

$$D/a^6 = (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\alpha)^2(\alpha-\gamma)^2(\beta-\delta)^2$$

$$
\begin{aligned}
&= {e_3}^4 - 4{e_3}^3(\alpha^3 + \beta^3 + \gamma^3 + \delta^3)\\
&\quad + 4{e_2}^3{e_3}^2\\
&\quad + 8e_2{e_3}^2\big(\alpha^3(\beta + \gamma + \delta) + \beta^3(\alpha + \gamma + \delta) + \gamma^3(\alpha + \beta + \delta) + \delta^3(\alpha + \beta + \gamma)\big)\\
&\quad + 16{e_3}^2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\delta^3 + \delta^3\alpha^3 + \alpha^3\gamma^3 + \delta^3\beta^3)\\
&\quad - 8{e_2}^3{e_3}^2\\
&\quad - 16{e_2}^2e_3\big(\alpha^3(\beta\gamma + \gamma\delta + \delta\beta) + \beta^3(\gamma\delta + \delta\alpha + \alpha\gamma) + \gamma^3(\alpha\beta + \delta\alpha + \delta\beta) + \delta^3(\alpha\beta + \beta\gamma + \alpha\gamma)\big)
\end{aligned}
$$

$$
\begin{aligned}
&-32e_2e_3\big(\alpha^3\beta^3(\gamma+\delta)+\beta^3\gamma^3(\delta+\alpha)+\gamma^3\delta^3(\alpha+\beta)+\delta^3\alpha^3(\beta+\gamma)+\alpha^3\gamma^3(\beta+\delta)\\
&\qquad\qquad+\delta^3\beta^3(\alpha+\gamma)\big)\\
&-64e_3(\alpha^3\beta^3\gamma^3+\beta^3\gamma^3\delta^3+\gamma^3\delta^3\alpha^3+\delta^3\alpha^3\beta^3)\\
&+16{e_2}^4e_4\\
&+32{e_2}^3e_4(\alpha^2+\beta^2+\gamma^2+\delta^2)\\
&+64{e_2}^2e_4(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\delta^2+\delta^2\alpha^2+\alpha^2\gamma^2+\delta^2\beta^2)\\
&+128e_2e_4(\alpha^2\beta^2\gamma^2+\beta^2\gamma^2\delta^2+\gamma^2\delta^2\alpha^2+\delta^2\alpha^2\beta^2)\\
&+256{e_4}^3
\end{aligned}
$$

$$
\begin{aligned}
={}&{e_3}^4-4{e_3}^3\cdot 3e_3\\
&+4{e_2}^3{e_3}^2\\
&+8e_2{e_3}^2(-2{e_2}^2+4e_4)\\
&+16{e_3}^2({e_2}^3-3e_2e_4+3{e_3}^2)\\
&-8{e_2}^3{e_3}^2\\
&-16{e_2}^2e_3\cdot(-2e_2e_3)\\
&-32e_2e_3({e_2}^2e_3+5e_3e_4)\\
&-64e_3({e_3}^3-3e_2e_3e_4)\\
&+16{e_2}^4e_4\\
&+32{e_2}^3e_4(-2e_2)\\
&+64{e_2}^2e_4({e_2}^2+2e_4)\\
&+128e_2e_4({e_3}^2-2e_2e_4)\\
&+256{e_4}^3
\end{aligned}
$$

$$
\begin{aligned}
={}&{e_3}^4-12{e_3}^4\\
&+4{e_2}^3{e_3}^2\\
&-16{e_2}^3{e_3}^2+32e_2{e_3}^2e_4\\
&+16{e_2}^3{e_3}^2-48e_2{e_3}^2e_4+48{e_3}^4\\
&-8{e_2}^3{e_3}^2\\
&+32{e_2}^3{e_3}^2\\
&-32{e_2}^3{e_3}^2-160e_2{e_3}^2e_4\\
&-64{e_3}^4+192e_2{e_3}^2e_4\\
&+16{e_2}^4e_4\\
&-64{e_2}^4e_4\\
&+64{e_2}^4e_4+128{e_2}^2{e_4}^2\\
&+128e_2{e_3}^2e_4-256{e_2}^2{e_4}^2\\
&+256{e_4}^3
\end{aligned}
$$

$$
\begin{aligned}
&= {e_3}^4 - 12{e_3}^4 + 48{e_3}^4 - 64{e_3}^4 \\
&\qquad + 4{e_2}^3{e_3}^2 - 16{e_2}^3{e_3}^2 + 16{e_2}^3{e_3}^2 - 8{e_2}^3{e_3}^2 + 32{e_2}^3{e_3}^2 - 32{e_2}^3{e_3}^2 \\
&\qquad + 32e_2{e_3}^2e_4 - 48e_2{e_3}^2e_4 - 160e_2{e_3}^2e_4 + 192e_2{e_3}^2e_4 + 128e_2{e_3}^2e_4 \\
&\qquad + 16{e_2}^4e_4 - 64{e_2}^4e_4 + 64{e_2}^4e_4 \\
&\qquad + 128{e_2}^2{e_4}^2 - 256{e_2}^2{e_4}^2 \\
&\qquad + 256{e_4}^3
\end{aligned}
$$

$$
= -27{e_3}^4 - 4{e_2}^3{e_3}^2 + 144e_2{e_3}^2e_4 + 16{e_2}^4e_4 - 128{e_2}^2{e_4}^2 + 256{e_4}^3
$$

となる。ゆえに、

$$
\begin{aligned}
D &= a^6(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\alpha)^2(\alpha-\gamma)^2(\beta-\delta)^2 \\
&= a^6(-27{e_3}^4 - 4{e_2}^3{e_3}^2 + 144e_2{e_3}^2e_4 + 16{e_2}^4e_4 - 128{e_2}^2{e_4}^2 + 256{e_4}^3) \\
&= a^6\left(-27\left(-\frac{d}{a}\right)^4 - 4\left(\frac{c}{a}\right)^3\left(-\frac{d}{a}\right)^2 + 144\left(\frac{c}{a}\right)\left(-\frac{d}{a}\right)^2\left(\frac{e}{a}\right) + 16\left(\frac{c}{a}\right)^4\left(\frac{e}{a}\right) - 128\left(\frac{c}{a}\right)^2\left(\frac{e}{a}\right)^2\right. \\
&\qquad\qquad \left. + 256\left(\frac{e}{a}\right)^3\right) \\
&= -27a^2d^4 - 4ac^3d^2 + 144a^2cd^2e + 16ac^4e - 128a^2c^2e^2 + 256a^3e^3
\end{aligned}
$$

を得る。

以上